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ABSTRACT

It is shown that the generalized equation of state of a warm relativistic electron fluid reduces in the small-m/e limit to the well-known double adiabatic law of collisionless magnetohydrodynamics.

An account has been given elsewhere of the theory of a so-called "warm relativistic electron fluid," based on a perturbation expansion in powers of the electron thermal velocity.[1] Here, it will be our purpose to examine the limiting form of this theory in the case where the unperturbed orbits are of vanishingly small gyroradius (it is essentially a slightly modified version of ordinary collisionless magnetohydrodynamics, as developed by Chew et al.[2]). We treat this case by formal expansion in powers of a second small parameter, which for convenience may be identified as m/e.

Let us use the notation (1.n) to denote Eq. (n) of Ref. 1. Equation (1.37), then, to leading order in m/e, reduces to the familiar equation of infinite conductivity,

$$F_{\mu\nu} w^{\nu} = 0 \quad , \tag{1}$$

or in the three-vector notation,

$$\vec{E} + \vec{v} \times \vec{B} = 0 \quad . \tag{2}$$

An immediate consequence of this is $\vec{E}\cdot\vec{B} = 0$, or

$$\det (F_{\mu\nu}) = c^{-2} (\vec{E}\cdot\vec{B})^2 = 0 \quad . \tag{3}$$

(All symbols are the same as in Ref. 1.) It must be emphasized that these and other relations presently to be derived therefrom are valid <u>only</u> to leading order in m/e.

In the local fluid rest frame, (2) reduces to $\vec{E} = 0$. Now let τ^{μ} denote the unit four-vector having, in this frame, the components $(\vec{B}/|\vec{B}|,0)$.

Relations manifestly satisfied by τ^μ in the rest frame (and therefore also, by relativistic invariance, in a general frame) include the following:

$$g_{\mu\nu}\,\tau^\mu\,\tau^\nu = 1 \quad , \tag{4}$$

$$\tau_\mu\,w^\mu = 0 \quad , \tag{5}$$

$$\tau_\mu\,F^{\mu\nu} = 0 \quad , \tag{6}$$

and

$$F_{\mu\nu}\,g^{\nu\lambda}\,F_{\lambda\rho} = \beta^2\,(\tau_\mu\tau_\rho - g_{\mu\rho} - c^{-2}\,w_\mu\,w_\nu) \quad , \tag{7}$$

where

$$\beta^2 = \frac{1}{2}\,F_{\mu\nu}\,F^{\mu\nu} = \vec{B}\cdot\vec{B} - c^{-2}\,\vec{E}\cdot\vec{E} \quad . \tag{8}$$

(In the rest frame, $\beta = |\vec{B}|$.)

At one point in the calculation that follows, accuracy to two orders in m/e will be needed. Let us say, then, that a relation is satisfied "strongly" if satisfied to more than one order in m/e. The strong relation corresponding to (6) is

$$\tau_\mu\,F^{\mu\nu} = \alpha w^\nu \quad , \tag{9}$$

where α is a small scalar invariant (of order m/e). The space projection of $\tau_\mu\,F^{\mu\nu}$ in the rest frame is $\vec{\tau} \times \vec{B} = 0$, and of course w^ν also has space projection zero in the rest frame.

A formula will also be needed for the time rate of change of β:

$$\beta D_s \beta = \frac{1}{2} F_{\mu\nu} D_s F^{\mu\nu}$$

$$= \frac{1}{2} F_{\mu\nu} w_\lambda \partial^\lambda F^{\mu\nu}$$

$$= - \frac{1}{2} F_{\mu\nu} w_\lambda (\partial^\mu F^{\nu\lambda} + \partial^\nu F^{\lambda\mu})$$

$$= \frac{1}{2} F_{\mu\nu} (F^{\nu\lambda} \partial^\mu w_\lambda + F^{\lambda\mu} \partial^\nu w_\lambda)$$

$$= \beta^2 (\Sigma - \Delta) \quad , \tag{10}$$

where

$$\Sigma = \tau_\mu \tau_\nu \partial^\mu w^\nu \quad , \tag{11}$$

$$\Delta = \partial_\mu w^\mu = - D_s \log n \quad . \tag{12}$$

Here, use has been made in turn of the Maxwell equation

$$\partial_\mu F_{\nu\lambda} + \partial_\nu F_{\lambda\mu} + \partial_\lambda F_{\mu\nu} = 0 \quad , \tag{13}$$

and of (1), (7), and (1.24). (We have also used $w_\lambda \partial_\mu w^\lambda = 0$. The four-vector w^μ is normalized to constant length.) Finally, using (12), let us rewrite (10) as

$$\Sigma = D_s \log (\beta/n) \quad . \tag{14}$$

We turn our attention now to the structure of the electron pressure tensor. Equation (1.38) to leading order reduces to

$$F_{\lambda\sigma}\,(g^{\lambda\mu}\,\Theta^{\sigma\nu} + g^{\lambda\nu}\,\Theta^{\sigma\mu}) = 0 \quad , \tag{15}$$

in which $\Theta^{\mu\nu}$ also satisfies condition (1.36). In the rest frame, it reduces further [here (1.36) is used] to

$$\Theta_{ij}\,B_{jk} = B_{ij}\,\Theta_{jk} \quad , \tag{16}$$

$$\Theta_{4i} = \Theta_{i4} = \Theta_{44} = 0 \quad . \tag{17}$$

A short calculation now shows that (16) can be satisfied only if Θ_{ij} is expressible as follows:

$$\Theta_{ij} = \Theta_{ji} = p_{\perp}\,\delta_{ij} + (p_{\parallel} - p_{\perp})\,\tau_i\tau_j \quad , \tag{18}$$

in which $p_{\perp}$ and $p_{\parallel}$ are scalar invariants (proper pressure components). This is, of course, simply the standard two-component pressure tensor of collisionless magnetohydrodynamics, first introduced in Ref. 2.

The last result is consistent with (1.3). The latter, to leading order, reduces to the statement that f in the rest frame is independent of the azimuth angle of $\vec{\zeta}$ in the plane transverse to $\vec{B}$, hence a function only of the invariants ξ and η, where

$$\xi = \tau_{\mu}\,\zeta^{\mu} \quad , \quad \eta^2 = \zeta_{\mu}\zeta^{\mu} - \xi^2 \quad . \tag{19}$$

In terms of f, the pressure components may be given as

$$p_\perp = \pi m \int_{-\infty}^{\infty} d\xi \int_0^{\infty} f\eta^3 \, d\eta \quad , \tag{20}$$

$$p_\parallel = 2\pi m \int_{-\infty}^{\infty} \xi^2 d\xi \int_0^{\infty} f\eta \, d\eta \quad . \tag{21}$$

The pressure tensor in a general frame is

$$\Theta^{\mu\nu} = p_\perp \, (g^{\mu\nu} + c^{-2} \, w^\mu \, w^\nu) + (p_\parallel - p_\perp) \, \tau^\mu \, \tau^\nu \quad . \tag{22}$$

It satisfies (strongly) the further relation

$$\tau_\mu \, F^{\mu\nu} \, \Theta_{\nu\lambda} = 0 \quad , \tag{23}$$

in consequence of (9) and (1.36).

The time evolution of the two proper pressure components is determined, essentially, by the interplay of the small terms in (1.38). We accordingly form the double scalar products of (1.38) in turn with the tensors $g_{\mu\nu}$ and $\tau_\mu \, \tau_\nu$. Both these operations are such as to annihilate the large term of order e/m, leaving only

$$nD_s \, (n^{-1} \, \Theta^\mu_\mu) = - \, 2\Theta^{\mu\nu} \, \partial_\mu \, w_\nu \quad , \tag{24}$$

$$\tau_\mu \, \tau_\nu \, nD_s \, (n^{-1} \, \Theta^{\mu\nu}) = - \, 2\tau_\mu \, \tau_\nu \, \Theta^{\nu\lambda} \, \partial_\lambda \, w^\mu \quad . \tag{25}$$

[Vanishing of the large term depends only on the strong relations (1.12), (1.31), and (23)]. If we now put in the explicit form of $\Theta^{\mu\nu}$ according to (22), we easily obtain

$$D_s\, p_\perp = p_\perp\, (\Sigma - 2\Delta) \quad , \tag{26}$$

$$D_s\, p_\parallel = -\, p_\parallel\, (2\Sigma + \Delta) \quad , \tag{27}$$

after a short calculation. Here, use has been made of (12) to eliminate n, and also of $\tau_\mu D_s\, \tau^\mu = 0$. If we also make use of (14), we can write this result as

$$D_s\, (p_\perp / n\beta) = D_s\, (\beta^2 p_\parallel / n^3) = 0 \quad , \tag{28}$$

and this, of course, is simply the relativistic form of the double-adiabatic law of Ref. 2.

Finally, I should like to correct a misstatement in Ref. 1. Only the second line of Eq. (1.82) is correct as a formula for $H_{ij\ell}$. The expression appearing in the first line is not $H_{ij\ell}$ itself but $H_{ij\ell}\, E_\ell$.

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